The isometry group of the bounded Urysohn space is simple

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Abstract

We show that the isometry group of the bounded Urysohn space is a simple group.

1 Introduction

The bounded Urysohn space $\mathbb{U}_1$ of diameter 1 is the (unique) complete homogeneous separable metric space of diameter 1 which embeds every finite metric space of diameter 1. It was shown in [1] that the isometry group of the (general) Urysohn space modulo the subgroup of bounded isometries is a simple group and it was widely conjectured (in particular by M. Rubin and J. Melleray) that the isometry group of the bounded Urysohn space is a simple group. We here prove this conjecture using the approach from [1]:

Theorem 1.1. *The isometry group of $\mathbb{U}_1$ is abstractly simple.*

Note that we cannot expect bounded simplicity as in the results in [1] as there are isometries of $\mathbb{U}_1$ with arbitrarily small displacement.

The proof relies on the properties of an abstract independence relation. We will continue to use the concepts introduced in [1], in particular the following notion of independence:

Definition 1.2. *We say that A and C are independent over B, written*

$$A \mathop{\downarrow}_{B} C,$$

if for all $a \in A, c \in C$ *with* $d(a,c) < 1$ *there is some* $b \in B$ *such that* $d(a,c) = d(a,b) + d(b,c)$.

We say that an automorphism $g \in \mathrm{Isom}(\mathbb{U}_1)$ *moves* almost maximally *if for all types* $\mathrm{tp}(a/X)$ *with* X *finite there is a realisation* b *with*

$$b \mathop{\downarrow}_{X} g(b).$$

Note that this definition of independence makes sense even if $B = \emptyset$ and hence this defines a stationary independence relation in the sense of [1]. The proof here follows the same lines as the proof in [1] and we will continue using notions from that paper. In the next section we will establish the following:

Proposition 1.3. *Let* $g \in \mathrm{Isom}(\mathbb{U}_1)$. *If* $d(a, g(a)) = 1$ *for some* $a \in \mathbb{U}_1$, *then a product of* 2^5 *conjugates of* g *moves almost maximally and hence any element of* $\mathrm{Isom}(\mathbb{U}_1)$ *can be written as the product of* 2^9 *conjugates of* g *and* g^{-1}.

Using the following observation, this proposition will then imply Theorem 1.1 exactly as in [1].

Lemma 1.4. *If* $g \in \mathrm{Isom}(\mathbb{U}_1)$ *is not the identity, then a product of conjugates of* g *moves some element by distance* 1.

Proof. Let $a \in \mathbb{U}_1$ be such that $d(a, g(a)) = k > 0$. Pick $b \in U_1$ with $d(a,b) = 1$ and a sequence of elements $a_0 = a, \ldots a_m = b$ with $d(a_{i-1}, a_i) = k, i = 1, \ldots m$. By homogeneity of $\mathbb{U}_1$ there are elements $h_i \in \mathrm{Isom}(\mathbb{U}_1), i = 1, \ldots m$ with $h_i(a) = a_{i-1}, h_i(g(a)) = a_i$. Then $g^{h_i}(a_{i-1}) = a_i$ and hence the product of these conjugates moves a to b. $\square$

2 Proof of the main result

For any finite set $X \subset \mathbb{U}_1, a \in \mathbb{U}_1$ we write $d(a, X) = \min\{d(a,x) \colon x \in X\}$ for the distance from a to A. We call $d(a, X)$ also the distance of the type $\mathrm{tp}(a/A)$. We put $G = \mathrm{Isom}(\mathbb{U}_1)$.

Lemma 2.1. *Let* $g \in G$ *be such that for some* $a \in \mathbb{U}_1$ *we have* $d(a, g(a)) = 1$. *Then for any finite set* A *there is some* x *with* $d(x, A) = 1$ *and* $d(x, g(x)) \geq 1/2$.

Proof. Clearly we may assume that $a \in A$. Put $Y = A \cup g^{-1}(A)$ and choose some b with $d(b,a) = 1/2$ and independent from Y over a. Then $d(g(b), A) \geq 1/2$ and since $d(a, g(a)) = 1$ we also have $d(g(b), a) = 1$. Therefore we have $d(b, g(b)) \geq 1/2$. Choose x with $d(x, Ab) = 1$ such that $d(x, g(b))$ is minimal. Since $d(g(b), Ab) \geq 1/2$, we have $d(x, g(b)) \leq 1/2$ and hence $d(x, g(x)) \geq 1/2$. □

Let $p = \mathrm{tp}(a/X)$ be a type over a finite set X. We say that $g \in G$ moves the type p *almost maximally* if there is a realisation x of p with $x \mathop{\downarrow}_X g(x)$ and it moves the type p by distance C if there is a realisation x of p with $d(x, g(x)) \geq C$.

Lemma 2.2. *Let $g \in G$ and $1 \geq d_0 \geq 0$ be such that g moves any type of distance d_0 almost maximally. Then any type of distance $d \leq d_0$ is moved almost maximally or by distance $1 - 2(d_0 - d)$.*

Proof. Let $p = \mathrm{tp}(x/X)$ be a type of distance $d \leq d_0$ and x' a realisation of p independent from $g^{-1}(X)$ over X (so $d(x', Xg^{-1}(X)) = d$). Put $p' = \mathrm{tp}(x'/Xg^{-1}(X))$ and let $q = p' + (d_0 - d)$ denote the prolongation of p' by $d_0 - d$.

By assumption on g, there is a realisation z of q which is moved almost maximally over $Xg^{-1}(X)$. Hence

$$z \mathop{\downarrow}_{Xg^{-1}(X)} g(z)$$

and by transitivity

$$z \mathop{\downarrow}_{X} g(z).$$

If $d(z, g(z)) = 1$ then for a realisation y of p' with $d(y, z) = d_0 - d$ we clearly have $d(y, g(y)) \geq 1 - 2(d_0 - d)$.

Otherwise we find some $b \in X$ such that

$$d(z, g(z)) = d(z, b) + d(b, g(z)).$$

Let y be a realisation of p' with $d(y, z) = d_0 - d$. Note that by definition of the prolongation we have

$$z \mathop{\downarrow}_{y} Xg^{-1}(X) \quad \text{and hence} \quad g(z) \mathop{\downarrow}_{g(y)} X.$$

Therefore

$$d(z, g(z)) = d(z, y) + d(y, b) + d(b, g(y)) + d(g(y), g(z))$$

and in particular

$$y \underset{X}{\bigcup} g(y).$$

□

Lemma 2.3. *Let $g \in G$. Then there exists some $h \in G$ such that $[g, h]$ has the following property for all d and C: if g moves all types of distance d almost maximally or by distance C, then $[g, h]$ moves all types of distance d almost maximally or by distance $2C$.*

Proof. As in [1] we may work in a countable model of the bounded Urysohn space. We build h by a 'back-and-forth' construction as the union of a chain of finite partial automorphisms. It is enough to show the following: let h' be already defined on the finite set U, let p be a type over X of distance d and assume that g moves all such types almost maximally or by distance C. Then h' has an extension h such that $[g, h]$ moves p almost maximally or by distance $2C$.

We may assume that X is contained in U. We denote by V the image of U under h'. Consider any realisation a of p independent from

$$Y = Ug^{-1}(U)g^{-1}(X)$$

and a realisation b of $h'(\mathrm{tp}(a/U))$ over V. Then we extend h' to $h : Uac \cong Vbg(b)$ where c is a realisation of $h^{-1}(\mathrm{tp}(g(b)/Vb))$ independent from $Xg(a)$. Then a is moved under $[g, h]$ to $g^{-1}(c)$. Since

$$c \underset{Ua}{\bigcup} g(a) \quad \text{and} \quad g(a) \underset{g(X)}{\bigcup} UX$$

we have $c \bigcup_{g(X)a} g(a)$, which means that

$$c \underset{a}{\bigcup} g(a) \quad \text{or} \quad c \underset{g(X)}{\bigcup} g(a).$$

The second case implies $g^{-1}(c) \bigcup_X a$, which implies our claim.

Since $d(a/Y) = d(a/U) = d$, our assumption about d and C implies that one of the following three cases occur:

Case 1. We find a and b as above with $d(a, g(a)) \geq C$ and $d(b, g(b)) \geq C$. By the above we may assume that $c \mathop{\downarrow}_a g(a)$. If $d(c, g(a)) = d(g^{-1}(c), a) = 1$, then $g^{-1}(c)$ and a are independent over the empty set and hence over X. Otherwise we have

$$d(g^{-1}(c), a) = d(c, g(a)) = d(c, a) + d(a, g(a)) = d(b, g(b)) + d(a, g(a)) \geq 2C.$$

Case 2. $a \mathop{\downarrow}_Y g(a)$: This implies $a \mathop{\downarrow}_X g(a)$. Since $g(a) \mathop{\downarrow}_{g(X)} X$ transitivity yields $a \mathop{\downarrow}_{g(X)} g(a)$. So from $c \mathop{\downarrow}_{ag(X)} g(a)$, then we get $c \mathop{\downarrow}_{g(X)} g(a)$ and hence $g^{-1}(c) \mathop{\downarrow}_X a$ as desired.

Case 3. $b \mathop{\downarrow}_V g(b)$: This implies $a \mathop{\downarrow}_U c$. As above we now get

$$c \mathop{\downarrow}_{g(X)} g(a) \quad \text{and hence} \quad g^{-1}(c) \mathop{\downarrow}_X a.$$

$\square$

By the results in [1] we now obtain:

Proposition 2.4. [1] *Let $g \in G$ be such that for some $a \in \mathbb{U}_1$ we have $d(a, g(a)) = 1$. Then every element of G is the product of 2^9 conjugates of g and g^{-1}.*

Proof. An iterated application of Lemma 2.3 to g yields isometries g_1, g_2, g_3,g_4 and g_5. Note that g_5 is a product of 2^5 conjugates of g and g^{-1}.

By Lemma 2.1 g moves every type with distance 1 by distance $\frac{1}{2}$. So g_1 moves every type of distance 1 almost maximally or by distance $2 \cdot 1/2 = 1$, hence almost maximally. Now Lemma 2.2 (with $d_0 = 1$) implies that g_1 moves every type of distance d almost maximally or by distance $1 - 2(1 - d) = 2d - 1$.

This implies that g_2 moves every type of distance d almost maximally or by distance $4d - 2$. So types of distance $d \geq \frac{3}{4}$ are moved almost maximally and

[1] We thank Adriane Kaïchouh and Isabel Müller for pointing out an error in an earlier version of the proposition.

using Lemma 2.2 with $d_0 = \frac{3}{4}$ we see that types of distance $d \leq \frac{3}{4}$ are moved almost maximally or by distance $1 - 2(\frac{3}{4} - d) = 2d - \frac{1}{2}$.

Now g_3 moves every type of distance d almost maximally or by distance $4d - 1$. So types of distance $d \geq \frac{1}{2}$ are moved almost maximally and using Lemma 2.2 with $d_0 = \frac{1}{4}$ we see that types of distance $d \leq \frac{1}{2}$ are moved almost maximally or by distance $1 - 2(\frac{1}{2} - d) = 2d$.

This implies that g_4 moves every type of distance d almost maximally or by distance $4d$. So types of distance $d \geq \frac{1}{4}$ are moved almost maximally and using Lemma 2.2 with $d_0 = \frac{1}{4}$ we see that types of distance $d \leq \frac{1}{4}$ are moved almost maximally or by distance $1 - 2(\frac{1}{4} - d) = 2d + \frac{1}{2}$.

So g_5 moves all types almost maximally. By Corollary 5.4 in [1], every element of G is a product of at most 2^4 conjugates of g_5 or its inverse. $\square$

Corollary 2.5. *Let $g \in G$. If there is $a \in \mathbb{U}_1$ with $d(a, g(a)) \geq 1/n$, then any element of G can be written as a product of at most $n \cdot 2^9$ conjugates of g and g^{-1}.*

References

[1] K. Tent and M. Ziegler. On the isometry group of the Urysohn space. *J. Lond. Math. Soc. (3)*, to appear.

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